I-NMC Sample Quesions Category X

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1. Albert and Bob were hired by the local authorities to paint the lamp-posts on both sides of a street. Albert arrived first and painted three posts on the south side before Bob arrived. Bob then pointed out that Albert was actually supposed to paint the north side, so Albert switched to the north side and started over, while Bob took over the south side.

   After Bob finished his side, he crossed the street and painted six more posts on Albert's side (the north side) to help complete the job. Given that both sides of the street had the same number of lamp-posts, the question is:

   Who painted more lamp-posts, and by how many more?

   Answer: ____________

2. The rule of an "obstacle course" specifies that at the nth obstacle a person has to roll a dice n times. If the sum of points in these n rolls is bigger than $2n$, the person is said to clear the obstacle. Now, at most how many obstacles can a person clear?

   Answer: ____________

3. Evaluate the integral

$$\int_{\infty}^{\infty} (x^3 + x^2 + x + 1)e^{-x^2}\,dx$$

   Answer: ____________

4. Let n be any natural number. Evaluate the following expression:

$$\left\lfloor \frac{n+2^0}{2^1} \right\rfloor + \left\lfloor \frac{n+2^1}{2^2} \right\rfloor + \left\lfloor \frac{n+2^2}{2^3} \right\rfloor + \dots$$

   Here $\lfloor x \rfloor$ is the floor of number x.

   Answer: ____________

5. Assume a and b are positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs of positive integers (a, b) such that $q^2 + r = 1997$.

   Answer: ____________

6. Consider a function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

   (a) $f(1) = 1$.
   (b) For all $x \in \mathbb{R}$, $f(x + 5) \geq f(x) + 5$.
   (c) For all $x \in \mathbb{R}$, $f(x + 1) \leq f(x) + 1$.

   Given $g(x) = f(x) - x + 1$, what is the value of $g(2002)$?

   Answer: ____________

7. Factorize $5^{1985} - 1$ into a product of three integers, all greater than 5^{100}.

Answer: ____________

8. A box starts with p white balls and q black balls, with an unlimited supply of additional black balls nearby. The following process repeats until only one ball remains: two balls are drawn uniformly at random —if they are the same color, both are discarded and a black ball is added from the pile, while if they differ, the black ball is discarded and the white ball is returned to the box. What is the probability that the final remaining ball is white? Explain your reasoning.

Answer: ____________

9. Find all pairs (a, b) of integers such that

$$1 + 2^a + 2^{2a+1} = b^2.$$

Answer: ____________