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Close Connections between the Methods of Laplace Transform, Quantum Canonical Transform, and Supersymmetry Shape-Invariant Potentials in Solving Schrödinger Equations

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For all commonly known solvable models of the Schrödinger equation, three different methods, Laplace transform, quantum canonical transform, and supersymetry shape-invariant potential, can be employed to obtain solutions. In contrast to the method of power expansion, these methods systematically reduce the Schrödinger equation to a first order differential equation, followed by integration to yield a closed form analytic solution. We analyze the correspondence between these methods and show: (1) All the commonly known solvable models can be divided into two classes. One corresponds to the hypergeometric equation and the other the confluent hypergeometric equation. For each class the sequential steps leading to the solutions are systematic and universal. (2) In both classes there is a precise correspondence between the steps of each method. Such a close connection offers insight into the long standing problem of explaining why solvable models are not abundant and why all these three analytical methods share a common set of solvable models.

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I. INTRODUCTION

In quantum mechanics there is a special class of continuous potential functions for which the Schrödinger equation can be solved analytically. These potential functions include

(1) $kx^2/2$, the harmonic oscillator,
(2) $A(x-1/x)^2$, the centripetal barrier potential [1–3],
(3) $A(e^{-2x}-2e^{-x})$, the Morse potential [1, 2],
(4) $-Ze^2/r$, the Coulomb potential,
(5) A/x^2, the centrifugal potential [4, 5],
(6) $-Ae^{-x}$, the central-force model of a deuteron [1],
(7) $-A\,\mathrm{sech}^2 x$, the modified Pöschl-Teller potential [1, 2],
(8) $A\csc^2 x$, the Pöschl-Teller potential [1, 6],
(9) $A(1-\coth x)$, the Hulthén potential [1],
(10) $A(1+\tanh x)$, the step potential [1],

where A is an arbitrary positive constant, x is the coordinate variable, and r in the Coulomb-potential model is the radius in spherical coordinates. Even though this class of solvable potentials contains only a limited number of function types, they carry the essential features of some important quantum systems, and their analytical solutions can provide much insight into these systems. In textbooks these potentials are commonly solved by the power expansion method. However, power expansion does not directly yield solutions in closed expressions which exhibit analytical properties or mathematical structures of the solutions.

Beside power expansion, three types of analytical method for solving the Schrödinger equations have been reported in the literature. (1) The iteration method, in which the excited states are constructed sequentially from the ground state by iterative operations using raising operators similar to the creation operator for the harmonic oscillator [7]. A recent development of this method is the supersymmetry shape-invariant potential [8–12]. (2) The method of quantum canonical transform, in which the Hamiltonian is reduced successively by a sequence of canonical transformations to a one-variable function. From the simple solution of the one-variable Hamiltonian the original solutions are obtained by the corresponding sequence of wavefunction transformations [13–16]. (3) The method of Laplace transform, in which variable changing and function substitution are used to reduce the coefficients of the equations to first-order polynomials, then by Laplace transform they become first-order differential equations and can be solved analytically [18–22]. Interestingly, all the four types of method (including power expansion) yield the same set of solvable potentials; no potential is found that can be solved by one method while not by the other methods. Therefore it would be interesting to consider the mathematical connection between these methods as a starting step to understand why this class of potential functions is special in being analytically solvable.

In this paper we show that the three seemingly different analytical methods (Laplace transform, quantum canonical transform, and supersymmetry shape-invariant potentials) have close correspondences between the respective steps that lead to the solutions. In Secs. II and III we give brief summaries for the methods of Laplace transform and quantum canonical transform. It will be shown that the three elementary quantum canonical transformations (point, similarity, and interchange transformations) correspond to variable changing, function substitution, and Laplace transform, respectively. In Sec. IV we show that all the solvable potentials listed above can be classified into only two classes, the confluent-hypergeometric class and the hypergeometric class, and within each class all the Schrödinger equations can be reduced to the same form. For each class we find the one-to-one correspondence between each step in the methods of quantum canonical transform and Laplace transform. In Sec. V we give a brief summary of the method of supersymmetry shape-invariant potentials and show that the shape-invariant condition and the factorization of the Hamiltonian into an iterative form correspond to the choice of variable changing plus the techniques of factoring out parts of the Hamiltonian and making substitutions in the method of Laplace transform. The analysis elucidates the connections between the three methods. From the analysis it also becomes clear why there are so few potential functions which are known to be solvable. For simplicity we set the mass $m = 1$ and $\hbar = 1$ in all the following sections.

II. THE METHOD OF LAPLACE TRANSFORM

The Laplace transform of a real function $f(x)$ is defined as

$$F(s) = \mathcal{L}[f(x)] = \int_0^\infty f(x)e^{-sx}dx, \tag{2.1}$$

and the inverse Laplace transform is [23, 24]

$$f(x) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_\Gamma F(s)e^{sx}ds, \tag{2.2}$$

where Γ is any simple closed contour containing all the poles of $F(s)$. The application of Laplace transform in solving differential equations is based on the following relations [25]. The first relation is

$$\mathcal{L}[f^{(n)}(x)] = s^n F(s), \tag{2.3}$$

when $f^{(k)}(0) = 0$ for $k = 0, 1, 2\cdots, n-1$. This identity can be proved by integration by parts, so does the similar one for the inverse Laplace transform

$$\mathcal{L}^{-1}[F^{(n)}(s)] = (-x)^n f(x). \tag{2.4}$$

The second relation is

$$\mathcal{L}\left[x^n f(x)\right] = (-1)^n F^{(n)}(s). \tag{2.5}$$

This identity can be proved by changing $x^n e^{-sx}$ to $(-d/ds)^n e^{-sx}$ and then taking $(-d/ds)^n$ out of the integral. Similarly one has for the inverse Laplace transform

$$\mathcal{L}^{-1}[s^n F(s)] = f^{(n)}(x). \tag{2.6}$$

From these relations it can be seen that the Laplace transform turns an equation with coefficient x^n into an n-th order differential equation. If $n \leq 1$, the differential equation can be solved simply by integration. If $n > 1$, the order of the equation after the Laplace transform is generally not reduced. This is why the Laplace transform is a popular method for solving ordinary differential equations with constant coefficients, but is not in favor for solving Schrödinger equations. However, with the help of variable changing and function substitution, the Laplace transform has been applied to the simple harmonic oscillator [18], the centripetal barrier potential [19], the Morse oscillator [20], the radial function of the hydrogen atom [21], and all other models listed in Sec. I [22].

In Sec. IV we present a new analysis to show that all the models listed in Sec. I can be divided into two classes, (A) the confluent-hypergeometric class and (B) the hypergeometric class. Models (1)–(6) listed in Sec. I belong to the class (A), which can be reduced to the confluent hypergeometric equation; whereas models (7)–(10) listed in Sec. I belong to the class (B), which can be reduced to the hypergeometric equation. The reduction is done by variable changing and universal forms of function substitution. The coefficients of the confluent hypergeometric equation are first-order polynomials. It can be readily solved by Laplace transform. This is not the case for the hypergeometric equation in which second-order polynomials appear in the coefficients. We solve the hypergeometric equation by inserting an additional function substitution between the Laplace transform and the inverse Laplace transform, which reduces the equation to a first-order differential equation after the inverse Laplace transform. By this technique models in the hypergeometric class are also solvable by Laplace transform.

III. THE METHOD OF QUANTUM CANONICAL TRANSFORM

In the method of quantum canonical transform one reduces the Hamiltonian from $H(p,x)$ to the one-variable form $H(J)$, where $J = -i\partial/\partial\theta$ is the action variable and θ is the angle variable, by using three elementary canonical transformations [13–16]. The eigenfunctions of $H(p,x)$ may be obtained from that of the action J by the wavefunction transformations associated with the elementary canonical transformations. The following table shows the definitions of the three elementary canonical transformations and their associated wavefunction transformations [14].

$$\text{interchange } \mathbf{I}: \quad (p,x) \to (x,-p), \qquad \psi_{\mathrm{I}}(x) = \int e^{ix\xi}\Psi(\xi)\,d\xi, \tag{3.1}$$

$$\text{similarity } \mathbf{S}: \quad (p,\,x) \to \left(p+f'(x),\,x\right), \qquad \psi_{\mathrm{S}}(x) = e^{if(x)}\Psi(x), \tag{3.2}$$

$$\text{point } \mathbf{P}: \quad (p,\,x) \to \left(\frac{1}{g'(x)}p,\,g(x)\right), \qquad \psi_{\mathrm{P}}(x) = \Psi(g^{-1}(x)), \tag{3.3}$$

where f' means df/dx and g^{-1} means the inverse function of g. $\psi_{\mathrm{I}}(x)$, $\psi_{\mathrm{S}}(x)$, and $\psi_{\mathrm{P}}(x)$ are the eigenfunctions before the interchange, similarity, and point transformations, respectively, if and only if $\Psi(x)$ is the eigenfunction after the transformations. Namely, they satisfy

$$H\,(p,\,x)\,\psi_{\mathrm{I}}(x) = E\psi_{\mathrm{I}}(x) \Leftrightarrow H\,(x,-p)\,\Psi(x) = E\Psi(x), \tag{3.4}$$

$$H\,(p,\,x)\,\psi_{\mathrm{S}}(x) = E\psi_{\mathrm{S}}(x) \Leftrightarrow H(p+f',\,x)\Psi(x) = E\Psi(x), \tag{3.5}$$

$$H\,(p,\,x)\,\psi_{\mathrm{P}}(x) = E\psi_{\mathrm{P}}(x) \Leftrightarrow H\left((1/g')p,\,g\right)\Psi(x) = E\Psi(x). \tag{3.6}$$

Note that the eigenvalues E are preserved under these transformations. These wavefunction transformations can be verified readily. For example, for the ψ_{I} in Eq. (3.1) $p\psi_{\mathrm{I}} = -i(d\psi_{\mathrm{I}}/dx)$ leads to $p\psi_{\mathrm{I}}(x) = \int e^{ix\xi}[\xi\Psi(\xi)]\,d\xi$ and integration by parts leads to $x\psi_{\mathrm{I}}(x) = \int e^{ix\xi}[-p\Psi(\xi)]\,d\xi$. This is the verification of Eq. (3.4).

In the procedure for reducing the Hamiltonians from $H(p,x) = p^2/2 + U(x)$ to $H(J)$, first a point transformation $\mathbf{P}$ with $x \to g(x)$ is used to simplify the potential $U(x)$. Then a similarity transformation $\mathbf{S}$ with $p \to p + f'(x)$ is used to bring in additional terms from $p^2/2$ to eliminate the simplified $U(x)$. With the right choices of $g(x)$ and $f(x)$ the Hamiltonian may become much simpler after these transformations. For example, it may become linear in x, which is called the x-linear form, or linear in p, which is called the p-linear form. The x-linear form can be brought to the p-linear form by an interchange transformation. If the Hamiltonian is a p-linear form, it corresponds to a first-order differential equation which can be solved by integration. In terms of quantum canonical transform such an integration corresponds to a linear transformation $\mathbf{L} \equiv \mathbf{P}^{-1}\mathbf{S}^{-1}$, which reduces a p-linear form to J. The operator transformation and the associated wavefunction transformation for the linear transformation are

$$\mathbf{L} \ : \ \left(\frac{1}{g'(x)}\left[p+f'(x)\right],\, g(x)\right) \to (J,\theta), \qquad \psi_{\mathrm{L}}(x) = e^{-if(x)}\Psi(g(x)), \tag{3.7}$$

which is the composite of $\mathbf{S}^{-1}$: $(p+f'(x),\, x) \to (p,x)$ and $\mathbf{P}^{-1}$: $([1/g'(x)]p,\, g(x)) \to (J,\theta)$. Instead of the Hamiltonian H, one can also choose to reduce $H-E$ to J, where E is the energy eigenvalue. Then the reverse sequence of the wavefunction transformations brings the trivial solution $\Psi = C$ (constant), which is in the null space of $J = -i\partial/\partial\theta$, to the eigenfunction of $H(p,x)$.

IV. CORRESPONDENCE BETWEEN QUANTUM CANONICAL TRANSFORM AND LAPLACE TRANSFORM

In this section we show that after appropriate variable changing $\xi = g(x)$, all the models in Sec. IV-1 can be reduced to the confluent hypergeometric equation by the substitution $\psi(\xi) = \xi^\beta v(\xi)$ and those in Sec. IV-2 can be reduced to the hypergeometric equation by the substitution $\psi(\xi) = (1+\xi)^\beta(1-\xi)^\gamma v(\xi)$. These two equations can further be reduced to first-order differential equations by Laplace or inverse Laplace transforms. Similarly, in the method of quantum canonical transform the point transformation in Eq. (3.3), which is just the variable changing $\xi = g(x)$, is used to simplify the potential. The similarity transformation in Eq. (3.2), which is just the function substitution $\psi(\xi) = \xi^\beta v(\xi)$ or $(1+\xi)^\beta(1-\xi)^\gamma v(\xi)$ by setting $e^{if(\xi)} = \xi^\beta$ or $(1+\xi)^\beta(1-\xi)^\gamma$, is used to eliminate the simplified potential. Finally the interchange transformation in Eq. (3.1), whose corresponding wavefunction transformation is the Fourier transform (similar to the Laplace transform), is used to bring the Hamiltonian to a p-linear form. After these transforms the equations can be solved by integration. In the following subsections we illustrate the one-to-one correspondence between the two methods with the following redefined parameters:

$$\alpha \equiv \sqrt{2A}, \qquad \lambda \equiv \sqrt{2|E|}. \tag{4.1}$$

IV-1. Confluent-hypergeometric class

This class contains the harmonic oscillator $U(x) = kx^2/2$, the centripetal barrier potential $U(x) = A(x-1/x)^2$, the Morse oscillator $U(x) = A(e^{-2x}-2e^{-x})$, the radial function of the hydrogen atom with the Coulomb potential $U(r) = -Ze^2/r$, the Bessel equation, the centrifugal potential $U(x) = A/x^2$, and the central-force model of deuteron $U(x) = -Ae^{-x}$. In applying the method of Laplace transform, one first reduces the Schrödinger equations of this class to the confluent hypergeometric equation,

$$\xi v'' + cv' - \left(\frac{\xi}{4} + d\right) v = 0, \tag{4.2}$$

by the variable changing $\xi = g(x)$ and the function substitution

$$\psi(\xi) = \xi^{\beta} v(\xi). \tag{4.3}$$

The coefficients of the confluent hypergeometric equation are linear in ξ, where $v' = dv/d\xi$. The function $g(x)$ and the constants c, d, β will be listed for each potential function in this class. There are always two choices of β that can reduce the Schrödinger equations to a confluent hypergeometric equation. One of the β will be stated first, and the other will be discussed later. For the harmonic oscillator the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi \tag{4.4}$$

can be simplified to Eq. (4.2) by changing the variable $\xi = \sqrt{k}x^2$ and making the substitution as Eq. (4.3), where $\beta = c/2 - 1/4 = 0$ (no function substitution is needed for this case) and $c = 1/2$, $d = -E/(2\sqrt{k})$. For the centripetal barrier potential the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} + A\left(x - \frac{1}{x}\right)^2 \psi = E\psi \tag{4.5}$$

can be simplified to Eq. (4.2) by changing the variable $\xi = \alpha x^2$ and making the substitution as Eq. (4.3), where $\beta = c/2 - 1/4$ and $c = (1/2)\sqrt{1 + 4\alpha^2} + 1$, $d = -\alpha/2 - E/(2\alpha)$. For the Morse oscillator the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} + A\left(e^{-2x} - 2e^{-x}\right)\psi = E\psi \tag{4.6}$$

can be simplified to Eq. (4.2) by changing the variable $\xi = 2\alpha e^{-x}$ and making the substitution as Eq. (4.3), where $\beta = c/2 - 1/2$ and $c = 2\lambda + 1$, $d = -\alpha$. For the radial function of the hydrogen atom the equation

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[2E + \frac{2Ze^2}{r} - \frac{l(l+1)}{r^2}\right] R = 0 \tag{4.7}$$

can be simplified to Eq. (4.2) by changing the variable $\xi = 2\lambda r$ and making the substitution as Eq. (4.3), where $\beta = c/2 - 1$ and $c = 2l + 2$, $d = -Ze^2/\lambda$. The Bessel equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(1 - \frac{\mu^2}{r^2}\right) R = 0, \tag{4.8}$$

where $\mu = 0, 1, 2, 3, \cdots$, can be simplified to Eq. (4.2) by changing the variable $\xi = 2ir$ and making the substitution as Eq. (4.3), where $\beta = c/2 - 1/2$ and $c = 2\mu + 1$, $d = 0$. For the centrifugal potential the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} + \frac{A}{x^2}\psi = E\psi \tag{4.9}$$

can be simplified to Eq. (4.2) by changing the variable $\xi = 2i\lambda x$ and making the substitution as Eq. (4.3), where $\beta = c/2$ and $c = 1 + \sqrt{1+4\alpha^2}$, $d = 0$. For the central-force model of the deuteron the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} - Ae^{-x}\psi = E\psi \tag{4.10}$$

can be simplified to Eq. (4.2) by changing the variable $\xi = 4i\alpha e^{-x/2}$ and making the substitution as Eq. (4.3), where $\beta = c/2 - 1/2$ and $c = 4\lambda + 1$, $d = 0$.

Then we apply the Laplace transform $\mathcal{L}[v(\xi)] = V(s)$ on the linear-coefficient differential equation (4.2). Since

$$\mathcal{L}[\xi v] = -V'(s), \qquad \mathcal{L}[v'] = sV(s), \tag{4.11}$$

$$\mathcal{L}[\xi v''] = -[s^2V(s)]' = -s^2V'(s) - 2sV(s), \tag{4.12}$$

it becomes the first order differential equation

$$\left(-s^2 + \frac{1}{4}\right)V'(s) + [(c-2)s - d]\,V(s) = 0. \tag{4.13}$$

The solution is

$$V(s) = C(s+1/2)^{c/2+d-1}(s-1/2)^{c/2-d-1}. \tag{4.14}$$

The inverse Laplace transform yields

$$v(\xi) = C\int_\Gamma (s+1/2)^{c/2+d-1}(s-1/2)^{c/2-d-1}e^{s\xi}ds \tag{4.15}$$

$$= Ce^{-\xi/2}{}_1F_1(c/2+d, c; \xi), \tag{4.16}$$

where the noncritical constant C may represent different values in different places and

$${}_1F_1(a, c; \xi) = C\int_\Gamma t^{a-1}(1-t)^{c-a-1}e^{t\xi}dt \tag{4.17}$$

is the confluent hypergeometric function [26]. Therefore the wavefunction is

$$\psi(\xi) = C\xi^\beta e^{-\xi/2}{}_1F_1\left(c/2+d, c; \xi\right), \tag{4.18}$$

where ξ, c, d, and $\beta = \beta(c)$ for each case are stated after Eqs. (4.4)–(4.10), respectively. Explicitly, they are $\beta(c) = 0$, $(1+\sqrt{1+4\alpha^2})/4$, λ, l, μ, $(1+\sqrt{1+4\alpha^2})/2$, 2λ, respectively. Changing c to $2-c$, one can reduce Eqs. (4.4)–(4.10) to the other confluent hypergeometric equation $\xi v'' + (2-c)v' - (\xi/4 + d)\,v = 0$ and obtain the second independent wavefunction

$$\psi(\xi) = C\xi^{\beta(2-c)}e^{-\xi/2}{}_1F_1\left(1-c/2+d, 2-c; \xi\right), \tag{4.19}$$

where $\beta(2-c) = 1/2$, $(1-\sqrt{1+4\alpha^2})/4$, $-\lambda$, $-l-1$, $-\mu$, $(1-\sqrt{1+4\alpha^2})/2$, -2λ, respectively. Note that here only for the harmonic oscillator $\beta = 1/2 > 0$, for the other cases one has

$\beta < 0$, hence except for the harmonic oscillator the second wavefunction in Eq. (4.19) diverges as $\xi \to 0$ and can be excluded. For the harmonic oscillator both solutions in Eqs. (4.18) and (4.19) are convergent, which are just

$$\psi = C\ {}_1F_1\left(-n, 1/2; \sqrt{k}x^2\right) = H_{2n}\left(\sqrt[4]{k}x\right) \tag{4.20}$$

and

$$\psi = Cx\ {}_1F_1\left(-n, 3/2; \sqrt{k}x^2\right) = H_{2n+1}\left(\sqrt[4]{k}x\right), \tag{4.21}$$

respectively [26]. Note that the first parameter in ${}_1F_1(a, c; \xi)$ must be $a = -n$ with $n = 0, 1, 2, 3, \cdots$, otherwise the confluent hypergeometric function would diverge as e^{ξ} when $\xi \to \infty$ [26]. The energy level for the cases in Eqs. (4.5)–(4.7) can be obtained by $a = c/2 + d = -n$ with $n = 0, 1, 2, 3, \cdots$. On the other hand, for the cases in Eqs. (4.8)–(4.10), where ξ are pure imaginary, e^{ξ} would not diverge and the solution in Eq. (4.18) remains finite for all E. Moreover, for the cases in Eqs. (4.8)–(4.10), one has $d = 0$, hence Eq. (4.18) is just

$$\psi = C\xi^{\beta}e^{-\xi/2}{}_1F_1\left(c/2, c; \xi\right) = C\xi^{\beta-\nu}J_{\nu}\left(\frac{\xi}{2i}\right) \tag{4.22}$$

with $\nu = (c-1)/2$ [26], where $J_{\nu}(x)$ is the νth order Bessel function. Namely, $R(r) = CJ_{\mu}(r)$ for the Bessel equation, $\psi(x) = Cx^{1/2}J_{(1/4+\alpha^2)^{1/2}}(\lambda x)$ for the centrifugal potential, and $\psi(x) = CJ_{2\lambda}(2\alpha e^{-x/2})$ for the central-force model of the deuteron.

For the confluent-hypergeometric class we use the centripetal barrier potential as an example to show the one-to-one correspondence of the steps above to that of quantum canonical transform. The Hamiltonian for the centripetal barrier potential is

$$H(p, x) = \frac{p^2}{2} + A\left(x - \frac{1}{x}\right)^2. \tag{4.23}$$

Corresponding to the variable changing $\xi = \alpha x^2$ and the function substitution $\psi(\xi) = \xi^{c/2-1/4}v(\xi)$ stated after Eq. (4.5), where $c = \sqrt{\alpha^2 + 1/4}+1$, we use here the point transform $\mathbf{P} : (p,\ x) \to \left(2\sqrt{\alpha\xi}p,\ \sqrt{\xi/\alpha}\right)$, which simplifies the potential $U(x) = A(x - 1/x)^2$ to $A(\xi/\alpha - 2 + \alpha/\xi)$ and changes $p^2/2$ to $2(\sqrt{\alpha\xi}p)^2$, and the similarity transform $\mathbf{S} : (p,\ \xi) \to (p - i(c/2 - 1/4)/\xi,\ \xi)$, which brings in additional terms from $2(\sqrt{\alpha\xi}p)^2$ to eliminate the last term $A\alpha/\xi$ in the simplified potential. Then $H - E$ becomes $2\alpha\, G(p, \xi)$, where

$$G(p, \xi) = \xi p^2 - icp + \frac{\xi}{4} + d \tag{4.24}$$

is linear in ξ just like Eq. (4.2) and $d = -\alpha/2 - \lambda^2/(4\alpha)$. Corresponding to the Laplace transform of Eq. (4.2), here we use the interchange transform $\mathbf{I} : (p, \xi) \to (\xi, -p)$ such that $G(p, \xi)$ becomes the p-linear form

$$-\left(\xi^2 + \frac{1}{4}\right)p - i(c-2)\xi + d, \tag{4.25}$$

just like Eq. (4.13). Eq. (4.25) can be reduced to a single dynamic variable J by the linear transform $\mathbf{L}$ in Eq. (3.7), where $1/g'(\xi) = -(\xi^2+1/4)$ and $f'(\xi) = [i(c-2)\xi - d]/(\xi^2+1/4)$. Starting from the constant solution $\Psi = C$ in the null space of J, the wavefunction before the linear transform is

$$\psi_{\mathrm{L}}(\xi) = e^{-if(\xi)}\Psi(g(\xi)) = C(\xi - i/2)^{c/2+d-1}(\xi + i/2)^{c/2-d-1}, \tag{4.26}$$

like Eq. (4.14). The wavefunction before the interchange transform is the Fourier transform of Eq. (4.26), which is the same as Eq. (4.15). Multiplying by $\xi^{c/2-1/4}$ and replacing ξ by αx^2, one obtains the wavefunction before the similarity and the point transforms, namely the wavefunction of the original $H(p,x)$, which is the same as Eq. (4.18). The one-to-one correspondence of the reduction steps shown above is not special to the centripetal barrier potential. It exists in all the models.

IV-2. Hypergeometric class

This class contains the two types of Pöschl-Teller oscillators $U(x) = -A\,\mathrm{sech}^2 x$ and $U(x) = A\csc^2 x$, the angular function of the hydrogen atom, the Hulthén potential $U(x) = A(1-\coth x)$, and the step potential $U(x) = A(1+\tanh x)$. In contrast to the cases in Sec. IV-1, in this class two function substitutions are needed (one before and one after the Laplace transform) to reduce the Schrödinger equations to a linear-coefficient equation, which becomes first-order and solvable after the inverse Laplace transform. First we use the Schrödinger equation of the Pöschl-Teller oscillator

$$-\frac{1}{2}\frac{d^2}{dx^2}\psi(x) - A\,\mathrm{sech}^2 x\psi(x) = E\psi(x) \tag{4.27}$$

as an example. By changing the variable

$$\xi = \tanh x, \tag{4.28}$$

Eq. (4.27) becomes

$$(1-\xi^2)\frac{d^2\psi}{d\xi^2} - 2\xi\frac{d\psi}{d\xi} + \left[l(l+1) - \frac{\lambda^2}{1-\xi^2}\right]\psi = 0, \tag{4.29}$$

where

$$l = \frac{1}{2}\left(-1+\sqrt{1+8A}\right). \tag{4.30}$$

Then using the substitution

$$\psi(\xi) = (1-\xi^2)^{\lambda/2}v(\xi) \tag{4.31}$$

to eliminate the $\lambda^2/(1-\xi^2)$ term, Eq. (4.29) becomes

$$(1-\xi^2)v'' - 2(\lambda+1)\,\xi v' + [l(l+1) - \lambda(\lambda+1)]v = 0. \tag{4.32}$$

The coefficient $1-\xi^2$ before $d^2\psi/d\xi^2$ in Eq. (4.29) can not be reduced to a first-order polynomial by any function substitution, because no matter what substitution one uses, both 1 and ξ^2 appear simultaneously before v'' and their orders differ by two. Hence we apply the Laplace transform $\mathcal{L}[v(\xi)] = V(s)$ directly on Eq. (4.32). Since

$$\mathcal{L}[\xi v'] = -[sV(s)]' = -sV'(s) - V(s). \tag{4.33}$$

$$\mathcal{L}[\xi^2 v''] = [s^2 V(s)]'' = s^2 V''(s) + 4sV'(s) + 2V(s). \tag{4.34}$$

the equation becomes

$$-s^2V'' + 2(\lambda-1)sV' + [s^2 - (\lambda+l)(\lambda-l-1)]V = 0. \tag{4.35}$$

Now only s^2 is left before V'', one more substitution

$$V(s) = s^\beta W(s) \tag{4.36}$$

can reduce the coefficients to first-order polynomials. Substituting Eq. (4.36) into (4.35), it becomes

$$-s^2W'' + 2(\lambda-\beta-1)sW' + [s^2 - (\beta-\lambda-l)(\beta-\lambda+l+1)]W = 0. \tag{4.37}$$

By choosing

$$\beta = \lambda + l \text{ or } \lambda - l - 1, \tag{4.38}$$

Eq. (4.37) becomes the linear-coefficient equation

$$-sW'' + 2(\lambda-\beta-1)W' + sW = 0. \tag{4.39}$$

Applying the inverse Laplace transform $\mathcal{L}^{-1}[W(s)] = w(\xi)$ on Eq. (4.39), since

$$\mathcal{L}^{-1}[sW''(s)] = [\xi^2 w(\xi)]' = 2\xi w + \xi^2 w' \tag{4.40}$$

one obtains the first order differential equation

$$(\xi^2-1)w' - 2(\beta-\lambda)\xi w = 0. \tag{4.41}$$

The solution is

$$w = (\xi^2-1)^{\beta-\lambda}. \tag{4.42}$$

The solution v is related to w by

$$W(s) = \mathcal{L}[w], \qquad v = \mathcal{L}^{-1}[V(s)] = \mathcal{L}^{-1}[s^\beta W(s)]. \tag{4.43}$$

The choice $\beta = \lambda + l > 0$ in Eq. (4.38) does not work for this case, because it implies from Eqs. (2.6) and $v = \mathcal{L}^{-1}[s^\beta W(s)]$ in Eq. (4.43)

$$v = \frac{d^\beta}{d\xi^\beta} w(\xi) = \frac{d^{\lambda+l}}{d\xi^{\lambda+l}} (\xi^2-1)^l, \tag{4.44}$$

and from Eq. (4.31)

$$\psi(\xi) = C(1-\xi^2)^{\lambda/2}\frac{d^{\lambda+l}}{d\xi^{\lambda+l}}(1-\xi^2)^l, \tag{4.45}$$

which is the Legendre polynomial. For $\xi = \tanh x$ and $-\infty < x < \infty$, this solution is non-square-integrable with respect to x. Therefore for this case one should choose $\beta = \lambda - l - 1 < 0$, and then Eqs. (4.42), (4.43) yield

$$W(s) = \mathcal{L}[w] = \int_{-1}^{1}(\xi^2-1)^{-l-1}e^{-s\xi}d\xi, \tag{4.46}$$

and

$$v(\xi) = \mathcal{L}^{-1}[s^{\lambda-l-1}W(s)] = C\int_{-1}^{1}(\xi'^{\,2}-1)^{-l-1}\left[\int_\Gamma s^{\lambda-l-1}e^{s(\xi-\xi')}ds\right]d\xi', \tag{4.47}$$

where $-1 \le \xi \le 1$ because $\xi = \tanh x$. Setting $1-\xi = 2\eta$ and $1-\xi' = 2t$, Eq. (4.47) equals

$$v(\eta) = C\int_0^1 t^{-l-1}(1-t)^{-l-1}\left[\int_\Gamma s^{\lambda-l-1}e^{-2s(\eta-t)}ds\right]dt. \tag{4.48}$$

Only when

$$\lambda - l = -n \tag{4.49}$$

and $n = 0, 1, 2, 3, \cdots$ is the integral in the brackets of Eq. (4.48) nonzero and has the residue $[-2(\eta-t)]^n/n!$. That is

$$\begin{aligned} v(\eta) &= C\int_0^1 t^{-l-1}(1-t)^{-l-1}(\eta-t)^n dt \\ &= C\,{}_2F_1(a,b,c;\eta), \end{aligned} \tag{4.50}$$

where

$${}_2F_1(a,b,c;\eta) = C\int_0^1 t^{a-c}(1-t)^{c-b-1}(\eta-t)^{-a}dt \tag{4.51}$$

is the hypergeometric function [27] and

$$a = -n, \quad b = 2l+1-n, \quad c = l+1-n. \tag{4.52}$$

Because $\eta = (1-\xi)/2$, Eq. (4.31) implies

$$\psi(\xi) = C(1-\xi^2)^{\lambda/2}{}_2F_1\left(a,b,c;\frac{1-\xi}{2}\right). \tag{4.53}$$

From Eqs. (4.30) and (4.49), the energy level is

$$E = -\frac{1}{2}\left[\frac{1}{2}\left(-1+\sqrt{1+8A}\right)-n\right]^2. \tag{4.54}$$

For the other Pöschl-Teller potential $A\csc^2 x$, the Schrödinger equation can also be reduced to Eq. (4.29), if the variable changing in Eq. (4.28) is replaced by $\xi = i\cot x$, and the solution is the same as Eq. (4.53).

The process above can also be applied to solve the angular function of the hydrogen atom. The Schrödinger equation for the Coulomb potential $U(r) = -Ze^2/r$ is

$$-\frac{1}{2}\nabla^2\psi - \frac{Ze^2}{r}\psi = E\psi. \tag{4.55}$$

By the method of separating variables one can show that the solution for this equation is

$$\psi(r,\theta,\phi) \;=\; R(r)\Theta(\theta)e^{i\mu\phi}, \tag{4.56}$$

where the radial function $R(r)$ satisfies Eq. (4.7) and the angular function $\Theta(\xi)$ with $\xi = \cos\theta$ satisfies

$$(1-\xi^2)\frac{d^2\Theta}{d\xi^2} - 2\xi\frac{d\Theta}{d\xi} + \left[l(l+1) - \frac{\mu^2}{1-\xi^2}\right]\Theta = 0, \tag{4.57}$$

$l = 0, 1, 2, 3, \cdots$, and $-l \le \mu \le l$, $\mu \in \mathbb{Z}$. Eq. (4.57) has the same appearance as Eq. (4.29). But unlike $\lambda = \sqrt{2|E|} > 0$ in Eq. (4.29), μ in Eq. (4.57) can be negative. For this case one can repeat the process from Eq. (4.31) to (4.45) by choosing $\beta = \mu + l > 0$, and the solution is the Legendre polynomial

$$\Theta(\xi) = C(1-\xi^2)^{\mu/2}\frac{d^{\mu+l}}{d\xi^{\mu+l}}(1-\xi^2)^l. \tag{4.58}$$

For $\xi = \cos\theta$ and $0 \le \theta \le \pi$, this solution is square-integrable with respect to θ. For this case the choice $\beta = \mu - l - 1 < 0$ does not work, because then the solution becomes $\Theta(\xi) = C(1-\xi^2)^{\mu/2}{}_2F_1\left(a, b, c; (1-\xi)/2\right)$, which diverges for $\theta = 0$ $(\xi = 1)$ and $\theta = \pi$ $(\xi = -1)$ when $\mu < 0$.

The Schrödinger equation for the Hulthén potential $A(1-\coth x)$ with $0 < x < \infty$ can be simplified to the hypergeometric equation

$$(1-\xi^2)v'' - \left[(a+b+1)\xi + c\right]v' - abv = 0, \tag{4.59}$$

where $a = -\beta+\gamma$, $b = -\beta+\gamma+1$, $c = 2(\beta+\gamma)$, by changing the variable $\xi = \coth x$ and making the substitution $\psi(\xi) = (1+\xi)^{-\beta}(1-\xi)^{\gamma}v(\xi)$ with $\beta = \sqrt{\alpha^2/2 + \lambda^2/4}$, $\gamma = \lambda/2$. For the step potential $A(1+\tanh x)$ with $0 < x < \infty$ the Schrödinger equation can also be reduced to Eq. (4.59), if the variable changing is replaced by $\xi = \tanh x$ and β, γ are changed to $\beta = i\lambda/2$, $\gamma = \sqrt{-\alpha^2/2 + \lambda^2/4}$. The Laplace transform on Eq. (4.59) yields

$$v(\xi) = C\,{}_2F_1\left(a,\, b,\, \frac{1}{2}(a+b+c+1);\, \frac{1-\xi}{2}\right), \tag{4.60}$$

where a, b, c are stated after Eq. (4.59). Therefore the wavefunction for the Hulthén potential and the step potential is

$$\psi = C(1+\xi)^{-\beta}(1-\xi)^{\gamma}{}_2F_1\left(-\beta+\gamma,\, -\beta+\gamma+1,\, 2\gamma+1;\, \frac{1-\xi}{2}\right). \tag{4.61}$$

The energy level can be obtained by $a = -\beta + \gamma = -n$ with $n = 0, 1, 2, 3, \cdots$, because otherwise ${}_2F_1(a, b, c; \xi)$ would diverge when $\xi \to \pm 1$ [27]. Changing γ to $-\gamma$, one obtains the second choice for reducing the Schrödinger equations to Eq. (4.59) and the second independent wavefunction

$$\psi = C(1+\xi)^{-\beta}(1-\xi)^{-\gamma}{}_2F_1\left(-\beta-\gamma,\ -\beta-\gamma+1,\ -2\gamma+1;\ \frac{1-\xi}{2}\right). \tag{4.62}$$

However, it diverges (because $1 - \xi \to 0$) as $x \to \infty$, hence it is excluded.

For this class we use the Pöschl-Teller oscillator $U(x) = -A\,\mathrm{sech}^2 x$ as an example to show the one-to-one correspondence of the steps above to the quantum canonical transform. The Hamiltonian for the Pöschl-Teller oscillator is

$$H(p, x) = \frac{p^2}{2} - A\,\mathrm{sech}^2 x. \tag{4.63}$$

Corresponding to the variable changing in Eq. (4.28), one uses here the point transform $\mathbf{P} : (p,\, x) \to \big((1-\xi^2)p,\, \tanh^{-1}\xi\big)$ to simplify the potential $-A\,\mathrm{sech}^2 x$ to $-A(1-\xi^2)$ and simplify $H - E$ to $[(1-\xi^2)/2]G(p, \xi)$, where

$$G(p, \xi) = p(1-\xi^2)p - l(l+1) + \frac{\lambda^2}{1-\xi^2} \tag{4.64}$$

like Eq. (4.29) and $l = (1/2)[-1+\sqrt{1+8A}]$ as Eq. (4.30). Corresponding to the substitution in Eq. (4.31), one uses then the similarity transform $\mathbf{S} : (p,\, x) \to \big(p + i\lambda\xi/(1-\xi^2),\, \xi\big)$ to bring in additional terms from $p(1-\xi^2)p$ to eliminate the last term $\lambda^2/(1-\xi^2)$ in Eq. (4.64), hence $G(p, \xi)$ is simplified to

$$p(1-\xi^2)p + 2i\lambda\xi p + \lambda(\lambda+1) - l(l+1), \tag{4.65}$$

like Eq. (4.32). The function in Eq. (4.65) can be factorized into $p\,F(p, \xi)$, where

$$F(p, \xi) = \left[1 - \left(\xi - \frac{i\beta}{p}\right)^2\right] p + 2i(l+1)\left(\xi - \frac{i\beta}{p}\right). \tag{4.66}$$

and $\beta = \lambda - l - 1$. Then the transform $\tilde{\mathbf{S}} : (p, \xi) \to (p,\, \xi + i\beta/p)$ can reduce $F(p, \xi)$ to

$$\big(1-\xi^2\big)\,p + 2i(l+1)\xi, \tag{4.67}$$

where $\tilde{\mathbf{S}}$ is the composite transform of $\mathbf{I}$ and $\mathbf{S} : (p, \xi) \to (p - i\beta/\xi,\, \xi)$ and $\mathbf{I}^{-1}$. Note that $\mathbf{I}^{-1}$: $(\xi, -p) \to (p, \xi)$ is the inverse of $\mathbf{I} : (p, \xi) \to (\xi, -p)$, and the wavefunction transform corresponding to $\tilde{\mathbf{S}}$ is the composite of the Fourier transform, the similarity transform (multiplying by $\xi^{\lambda-l-1}$), and the inverse Fourier transform, just like the composite of $W(s) = \mathcal{L}[w]$ and $v(\xi) = \mathcal{L}^{-1}[s^{\lambda-l-1}W(s)]$ in Eqs. (4.46) and (4.47). The function in Eq. (4.67) is linear in p and can be reduced to J by the linear transform $\mathbf{L}$ in Eq. (3.7), where $1/g'(\xi) = 1 - \xi^2$ and $f'(\xi) = 2i(l+1)\xi/(1-\xi^2)$. Starting from the constant solution $\Psi = C$ in the null space of J, the wavefunction before the linear transform is

$$\psi_{\rm L}(\xi) = e^{-if(\xi)}\Psi(g(\xi)) = C(\xi^2 - 1)^{-l-1}, \tag{4.68}$$

which is the same as Eq. (4.42). Then through Eqs. (4.46) and (4.47) one obtains the wavefunction before the $\tilde{\mathbf{S}}$ transform. Finally, multiplying by $(1-\xi^2)^{\lambda/2}$ and replacing ξ by $\tanh x$, one obtains the wavefunction before the similarity and point transforms, namely the wavefunction of the original $H(p,x)$, which is the same as Eq. (4.53). In this example we factorize $H - E$ into some nonconstant parts and integrate only the part that has the same null space. This technique is analogous to the use of integration factors in classical integration, where a differential form is integrated by factorizing out or multiplying an integration factor first.

V. CORRESPONDENCE BETWEEN SUPERSYMMETRY SHAPE-INVARIANT POTENTIALS AND LAPLACE TRANSFORM

In the method of supersymmetry shape-invariant potential, the n-th state wavefunction ψ_n can be obtained from the ground-state wavefunction ψ_0 with the ground-state energy adjusted to zero,

$$H\psi_0 = \left[-\frac{1}{2}\frac{d^2}{dx^2} + U(x)\right]\psi_0 = 0. \tag{5.1}$$

Define $\xi(x) \equiv -\psi_0'/\psi_0$ and

$$\mathbf{A} \equiv \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \xi\right), \tag{5.2}$$

$$\mathbf{A}^\dagger \equiv -\frac{1}{\sqrt{2}}\left(\frac{d}{dx} - \xi\right). \tag{5.3}$$

From Eq. (5.1) one has

$$U(x) = \frac{1}{2}\frac{\psi_0''}{\psi_0} = \frac{1}{2}\left(\xi^2 - \xi'\right). \tag{5.4}$$

Define its supersymmetric partner as

$$U^+(x) \equiv \frac{1}{2}\left(\xi^2 + \xi'\right). \tag{5.5}$$

The Hamiltonian can be written as

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + U(x) = \mathbf{A}^\dagger\mathbf{A}, \tag{5.6}$$

$$H^+ \equiv -\frac{1}{2}\frac{d^2}{dx^2} + U^+(x) = \mathbf{A}\mathbf{A}^\dagger. \tag{5.7}$$

If ψ_n^+ is an eigenfunction of H^+ with eigenvalue E_n^+, then $\mathbf{A}^\dagger\psi_n^+$ is an eigenfunction of H with the same energy,

$$H(\mathbf{A}^\dagger\psi_n^+) = \mathbf{A}^\dagger\mathbf{A}(\mathbf{A}^\dagger\psi_n^+) = \mathbf{A}^\dagger H^+\psi_n^+ = E_n^+(\mathbf{A}^\dagger\psi_n^+). \tag{5.8}$$

Namely, the eigenstates and eigenvalues of H are related to those of H^+ by

$$\psi_{n+1} = (E_n^+)^{-1/2}\mathbf{A}^\dagger\psi_n^+, \qquad E_{n+1} = E_n^+. \tag{5.9}$$

Suppose that U and U^+ are shape invariant. Namely, for the potential $U = U(x; c_0)$ where c_0 is a parameter of U, the supersymmetric partner U^+ satisfies

$$U^+ = U(x; c_1) + R(c_1), \tag{5.10}$$

where $c_1 = f(c_0)$ (function of c_0) and the remainder $R(c_1)$ is independent of x. Then the n-th eigenstate and eigenvalue of H^+ are related to those of H by

$$\psi_n^+ = \psi_n(x; c_{n+1}), \qquad E_n^+ = E_n + R(c_{n+1}), \tag{5.11}$$

where $c_k = f^k(c_0)$ (i.e., the function f applied k times). Starting from the ground-state ψ_0 and the ground energy E_0 and using Eqs. (5.9) and (5.11) iteratively, one obtains the n-th eigenfunction ψ_n and the n-th eigenvalue

$$E_n = E_0 + \sum_{k=1}^{n} R(c_k). \tag{5.12}$$

From the shape-invariant condition of $U = (\xi^2 - \xi')/2$ and $U^+ = (\xi^2 + \xi')/2$ one can postulate that ξ' is a quadratic function of ξ,

$$\xi' \;=\; c_2\xi^2 + c_1\xi + c_0, \tag{5.13}$$

namely

$$\xi' \;=\; c_2\left(\xi + \frac{c_1}{2c_2}\right)^2 + \left(c_0 - \frac{c_1^2}{4c_2}\right), \tag{5.14}$$

where c_0, c_1, and c_2 are constants. Assume $c_1 = 0$ for simplicity, it becomes

$$\xi' = c_2\xi^2 + c_0. \tag{5.15}$$

That is

$$\xi = a\tan(\sqrt{c_0c_2}\,x + C) + b\cot(\sqrt{c_0c_2}\,x + C') \qquad \text{when } c_0c_2 > 0, \tag{5.16}$$

or

$$\xi = a\tanh(\sqrt{|c_0c_2|}\,x + C) + b\coth(\sqrt{|c_0c_2|}\,x + C') \qquad \text{when } c_0c_2 < 0, \tag{5.17}$$

where the coefficients a, b, and C, C' are constants. This covers all the cases in the hypergeometric class, where Eq. (5.16) is the variable changing for the Pöschl-Teller potential $U(x) = A\csc^2 x$; and Eq. (5.17) is that for the other Pöschl-Teller potential $-A\,\text{sech}^2 x$ as well as the Hulthén potential $A(1-\coth x)$ and the step potential $A(1+\tanh x)$. Furthermore, if we assume that only one of c_0, c_1, or c_2 in Eq. (5.13) is nonzero, Eq. (5.13) becomes

$$\xi' = c_0 \quad \text{or} \quad \xi' = c_1\xi \quad \text{or} \quad \xi' = c_2\xi^2. \tag{5.18}$$

That is

$$\xi = c_0 x + C \quad \text{or} \quad Ce^{c_1 x} \quad \text{or} \quad 1/(c_2 x + C). \tag{5.19}$$

This covers all the cases in the confluent-hypergeometric class, where the first one $\xi = c_0 x + C$ is for the harmonic oscillator $U(x) = kx^2/2$, the centripetal barrier potential $U(x) = A(x-1/x)^2$, the radial function of the hydrogen atom with the Coulomb potential $U(r) = -Ze^2/r$, the Bessel equation, and the centrifugal potential $U(x) = A/x^2$ [28]; the second one $\xi = Ce^{c_1 x}$ is for the Morse oscillator $U(x) = A(e^{-2x} - 2e^{-x})$ and the central-force model of the deuteron $U(x) = -Ae^{-x}$.

Changing variable to ξ, the Schrödinger equation $H\psi = E\psi$, namely

$$\frac{d^2\psi}{dx^2} + (\xi' - \xi^2 + 2E)\psi = 0, \tag{5.20}$$

becomes

$$\xi'^2 \frac{d^2\psi}{d\xi^2} + \xi'' \frac{d\psi}{d\xi} + (\xi' - \xi^2 + 2E)\psi = 0. \tag{5.21}$$

Restricting to the shape-invariant condition, $\xi' = c_2\xi^2 + c_0$ ($\xi'' = 2c_2\xi\xi'$), Eq. (5.21) becomes

$$(c_2\xi^2 + c_0)^2 \frac{d^2\psi}{d\xi^2} + 2c_2\xi(c_2\xi^2 + c_0)\frac{d\psi}{d\xi} + \left[(c_2 - 1)\xi^2 + c_0 + 2E\right]\psi = 0. \tag{5.22}$$

The coefficients of Eq. (5.22) are polynomials of ξ with order ≤ 4 just like Eq. (4.29), which is the Schrödinger equation for the Pöschl-Teller oscillator after the variable changing in the Laplace-transform method. If $c_0 \neq 0$, Eq. (5.22) can be reduced by the substitution $\psi = (c_2\xi^2 + c_0)^\beta v(\xi)$ to

$$(c_2\xi^2 + c_0)\frac{d^2 v}{d\xi^2} + 2c_2\xi(2\beta + 1)\frac{dv}{d\xi} + cv = 0, \tag{5.23}$$

where $\beta = \sqrt{1 + 2Ec_2/c_0}/(2c_2)$ and $c = 1 + 2E/c_0 + \sqrt{1 + 2Ec_2/c_0}$. The coefficients of Eq. (5.23) are polynomials of order ≤ 2 just like Eq. (4.32), which is the Schrödinger equation (4.29) after the function substitution in the Laplace-transform method. If $c_0 = 0$ (namely $\xi' = c_2\xi^2$), Eq. (5.22) can be reduced by the variable changing $\eta = 1/\xi$ to

$$c_2^2\eta^2 \frac{d^2\psi}{d\eta^2} + \left(c_2 - 1 + 2E\eta^2\right)\psi = 0. \tag{5.24}$$

Moreover, for the condition $\xi' = c_0$ ($\xi'' = 0$), Eq. (5.21) becomes

$$c_0^2 \frac{d^2\psi}{d\xi^2} + (-\xi^2 + c_0 + 2E)\psi = 0, \tag{5.25}$$

and for $\xi' = c_1\xi$ ($\xi'' = c_1\xi'$) Eq. (5.21) becomes

$$c_1^2\xi^2 \frac{d^2\psi}{d\xi^2} + c_1^2\xi \frac{d\psi}{d\xi} + (c_1\xi - \xi^2 + 2E)\psi = 0. \tag{5.26}$$

The coefficients of Eqs. (5.24)–(5.26) are all polynomials of order ≤ 2. The Equations (5.23)–(5.26) can easily be reduced to linear-coefficient ones (order ≤ 1) by function substitution or by factoring out some functions of ξ and $d/d\xi$ as shown in Sec. IV.

VI. SUMMARY

In this paper we have done a thorough analysis to show the connections between the methods of Laplace transform, quantum canonical transform, and supersymmetry shape-invariant potentials in solving the Schrödinger equation. The former two methods give closed-form analytic solutions for all commonly known solvable models, whereas the latter gives the same analytic solutions by iterative operations. Through the analysis it is found that all the models can be divided into two classes, one corresponding to the hypergeometric equation and the other the confluent hypergeometric equation, and the steps leading to the solutions are systematic and universal. The requirement of bringing the Schrödinger equation to the linear-coefficient form in the Laplace-transform method corresponds nicely to the requirement of p-linear form in the method of quantum canonical transform as well as the shape-invariant condition in the method of supersymmetry shape-invariant potential. These tight requirements explain why there are so few solvable models, and the connections between them explain why all these three existing analytic methods share the same set of solvable models. In comparison to the commonly adopted power-expansion method, the analysis provides a rational and insightful approach to solving the Schrödinger equation.

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